Int J Theor Phys (2010) 49: 1612–1621
DOI 10.1007/s10773-010-0343-0

Parameter Region for Existence of Non-classical Solitons

M. Aguero · F. Ongay · J. Socorro

Received: 19 January 2010 / Accepted: 31 March 2010 / Published online: 15 April 2010

Abstract We study a pure nonlinear model of a chain formed by particles that are linked each other by an enharmonic type. This lattice nonlinear Klein–Gordon model is subsequently studied in its continuum version. We use the dynamical systems approach for analyzing the properties of the non-classical structures that support the model. Several non-classical structures like peakons, kink compactons and crodwon or bubble compactons are generated along the chain for the specific region of the parameter space. It is shown that the phase space trajectories are nonclassical curves and show unexpected behaviors. The first type of phase transition in the parametric space occurs when the number of centers and saddles changes while the main phase state parameter becomes critical.

Keywords Non-classical solitons · Crowdons · Compactons · Peakons · Phase portrait

1 Introduction and Mathematical Model

In several recent investigation the Hamiltonian for a chain of masses with pure nonlinear potential piece has caught a great attention [1, 2]. This interest has been rising towards the phenomenon of soliton like localization in the continuum versions of several important discrete nonlinear lattices. This model is a fully nonlinear one. As pointed out in the work [3], a crucial difference between the linear or weakly nonlinear models versus the fully nonlinear models is that the latter admit nonanalytic solutions or non classical soliton solutions. While substantial progress was made on the analytical and numerical edge, it has happened only recently that despite the emerging of discrete breathers along these lattices [4], it is suggested that this model could also admit other types of nonclassical solutions like peakons

M. Aguero (✉) · F. Ongay
Universidad Autonoma del Estado de Mexico, Instituto Literario 100 Toluca, Facultad de Ciencias, Estado de México 50000, Mexico
e-mail: maaguerog@uaemex.mx

J. Socorro
Universidad de Guanajuato, Apartado E-143, CP 37150, Leon GTO, Mexico

and crowdons or gray compactons for instance [5]. As is know the concept of compactons gained a huge recognition after the pioneering work done by Rosenau and Hyman [6]. We will address this issue here by investigating the dynamics on traveling waves solutions arising in the continuum limit of the pure nonlinear Klein-Gordon lattice model. Our ensuing analysis in this work is carried out on the basis of the specific model proposed in [7, 8]. The nonlinear wave equation that we will study is of the following form:

$$\ddot{\theta}_n + \kappa\left[(\theta_n - \theta_{n-1})^3 + (\theta_n - \theta_{n+1})^3\right] + \left(\alpha\theta_n - \beta\theta_n^3\right) = 0 \tag{1}$$

which appears from the Hamiltonian

$$H = \sum_{n=1}^{\infty}\left[\frac{1}{2}\dot{\theta}_n^2 + U(\theta_n - \theta_{n-1}) + V(\theta_n)\right] \tag{2}$$

where it has been considered a lattice of identical particles for which the neighboring interactions are modeled by the pure nonlinear potential

$$U(\theta_n - \theta_{n-1}) = \frac{1}{4}\kappa(\theta_n - \theta_{n-1})^4, \tag{3}$$

and when the system is subjected to external forces given by the potential V at the site "n"

$$V(\theta_n) = \frac{\alpha\theta_n^2}{2} - \frac{\beta\theta_n^4}{4}. \tag{4}$$

The approach to deriving analytical solutions was done explicitly for the case of traveling waves. Thus was postulated the emerging of solitary waves that travel along the chain and further was identified the range of parameter values for which these solutions exist. Several solutions was presented in the work [9] and here we will study their dynamical properties taking in consideration the phase portrait and making conclusions concerning the bifurcations and mappings of phase trajectories with the corresponding analytical solutions. We will first obtain the critical points and second the phase trajectories of the studied system. The critical points of saddle type play a crucial role for emerging non-classical solutions. The closed orbit passing through the saddle is called a homoclinic orbit. Commonly, the saddle are points doubly connected, but in our case since we have compact like solutions, truncated homoclinic orbits will be observed. Our model does not support damping terms too, then the oscillating analog "particle" will not be attracted down to bottoms i.e. centers. Also we have not here periodic excitations, that together with the dumping terms could eventually lead our system to exhibit chaotic behaviors [10]. But as is known the merely existence of homoclinic orbits is closely related to fundamental chaotic motion.

In the next section we present the equation of motion for the pure nonlinear model of joined particles along the chain. We present also the general phase portrait for the phase trajectories of the system. The Sect. 3 is devoted to the study of special phase trajectories and their corresponding non-classical solutions. Different types of bifurcations is presenting in Sect. 3.1 and finally in the last section some general discussions and conclusions are presented.

2 Equation of Motion and Phase Portrait

Suppose the field θ varies slowly from one site to another, and $2\beta \ll \kappa$, thus one can use the continuum approximation that was successfully employed in [7, 8, 11] and (1) transforms to the next nonlinear differential equation

$$\ddot{\theta} - \kappa \left[3 \left(\frac{\partial \theta}{\partial x} \right)^2 \frac{\partial^2 \theta}{\partial x^2} \right] + \left(\alpha\theta - \beta\theta^3 \right) = 0. \tag{5}$$

We discuss the parameter region for α, β, κ for the important case of traveling waves solutions of the nonlinear differential equation (5). Subsequently, making the standard change of variables for traveling waves in (5), $s = x - ut \to \theta(s) = \theta(x - ut)$, with arbitrary velocity value u, we obtain the new nonlinear equation of motion

$$u^2 \frac{d^2\theta}{ds^2} - 3\kappa \left(\frac{d\theta}{ds} \right)^2 \frac{d^2\theta}{ds^2} + \alpha\theta - \beta\theta^3 = 0, \tag{6}$$

where it is assumed that $u \neq 0$, $\kappa, \beta > 0$ and $\alpha \geq 0$ are constants.

Next, we convert this differential equation into a system of first order differential equations taking $y = \frac{d\theta}{ds} = \theta'$

$$\theta' = y, \qquad y' = \frac{\beta\theta^3 - \alpha\theta}{u^2 - 3\kappa y^2}. \tag{7}$$

The state or phase space is the nonbounded strip $-\infty < \theta < +\infty$, $-\frac{u}{\sqrt{3\kappa}} < y < \frac{u}{\sqrt{3\kappa}}$.

Remark 1 If $y \to \pm\frac{u}{\sqrt{3\kappa}}$, $y' \to \pm\infty$. This implies that the trajectories of the system (7) tend to have vertical slopes when y approaches—in finite time—the border of the strip (except when $\beta\theta^3 - \alpha\theta \to 0$).

The classification of equilibria is determined by whether α and β is positive or negative. The constant $\beta \neq 0$ will not influence the number of real roots of $y' = 0$ but α does. We mainly discuss the number of multiplicity of roots and discriminate the types of equilibria. The system (7) has three equilibria

$$(0, 0), \qquad \left(0, \pm\sqrt{\frac{\alpha}{\beta}} \right).$$

Remark 2 If $\alpha = 0$, these equilibria reduce to only one equilibrium.

Proposition *The function*

$$F(\theta, y) = \alpha\theta^2 - \frac{\beta}{2}\theta^4 + u^2 y^2 - \frac{3\kappa}{2} y^4$$

is a first integral of the system (7).

Proof Assume $(\theta(s), y(s))$ is a solution of system (7), then

$$\frac{d}{ds} F(\theta(s), y(s)) = \frac{\partial F}{\partial \theta}(\theta(s), y(s))\, \theta'(s) + \frac{\partial F}{\partial y}(\theta(s), y(s))\, y'(s)$$

$$= \left(2\alpha\theta(s) - 2\beta\theta(s)^3\right) y(s) + \left(2u^2 y(s) - 6\kappa y(s)^3\right) \frac{\beta\theta(s)^3 - \alpha\theta(s)}{u^2 - 3\kappa y(s)^2}$$
$$= 0. \qquad \square$$

Thus F is constant on the system's trajectories and hence they are contained on the level curves of F. We have three possible scenarios when $\alpha > 0$ and one when $\alpha = 0$.

Next, we will show different aspects of phase trajectories. The system trajectories are the part of the level curves contained in the region

$$-\frac{u}{\sqrt{3\kappa}} < y < \frac{u}{\sqrt{3\kappa}}.$$

Note that there are three equilibria when $\alpha > 0$ and only one when $\alpha = 0$. In the former case, the equilibrium $(0, 0)$ is a center and the equilibria $(0, \pm\sqrt{\frac{\alpha}{\beta}})$ are saddles. Trajectories that reach the border of the strip correspond with solutions of (6) defined on bounded or semibounded intervals except in certain cases when $\frac{\alpha^2}{\beta} = \frac{u^4}{3\kappa}$ that we will show later. If we identify the border of the strip as it was a cylinder, in a natural way we can easily extend in continuous way the solutions of (6) to piecewise differentiable solutions defined in the whole real line. This can be doing continuing trajectories throughout the paste line of the cylinder in a unique way, except in few cases, that will be explained later, when $\alpha > 0$. Due to the symmetry of the phase portraits, almost all extended solutions becomes periodic.

3 Special Trajectories and Solutions

We will examine the trajectories of the system (7) paying attention to those formed with the separatrices of the saddle equilibria, beside them and the equilibria themselves, all the other trajectories are cycles in the cylinder and correspond to periodic solutions. The trajectories together with their corresponding solutions will be shown and explained in Figs. 1 to 9, the variant parameter is α, the others are fixed at $\beta = 1$, $u = 2$ and $\kappa = 3$.

3.1 Phase Trajectories for $\alpha = 0$

This case corresponds to the analysis of the equation of motion under the trivial boundary condition i.e. to the case when $s \to \pm\infty, \theta \to 0, \theta_s \to 0$. It was found [9] the analytical solution for this case in terms of the following transcendental equation

$$\begin{aligned}
\xi\sqrt{\frac{a\sqrt{\sigma}}{2}}(^{-}s - s_0) = {} & -\mathrm{Arctan}\left[-1 + \sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}}\right] - \mathrm{Arctan}\left[1 + \sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}}\right] \\
& - \frac{1}{4}\mathrm{Ln}\left[\left(-1 + \sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}} - \frac{\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}\right)^2\right] \\
& + \frac{1}{4}\mathrm{Ln}\left[\left(1 + \sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}} + \frac{\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}\right)^2\right] \\
& - 4\sqrt{\frac{1+\sqrt{1-\sigma\theta^4}}{2\sqrt{\sigma}\theta^2}}
\end{aligned}$$

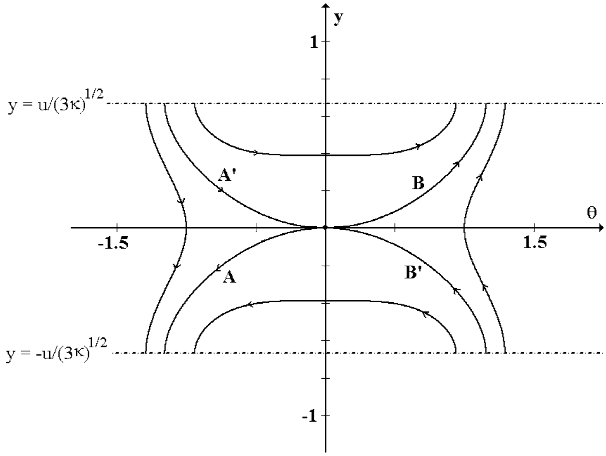


Fig. 1 Trajectories of the system (7) with $\alpha = 0$. Arcs **A** and **A**$'$ are one trajectory, arcs **B** and **B**$'$ are other single trajectory

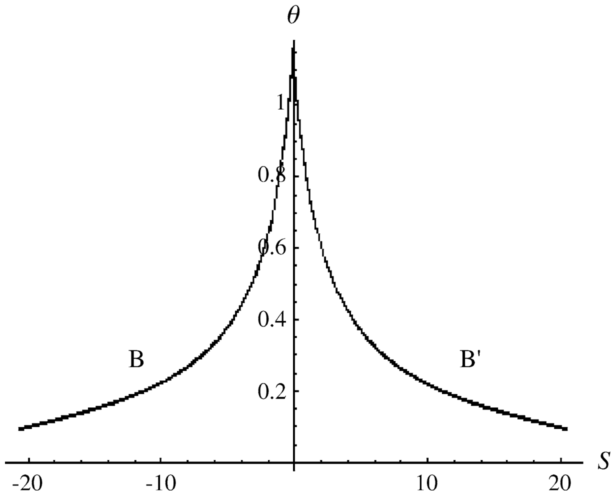


Fig. 2 Numerical representation of the solution corresponding to trajectory B of Fig. 1. Solution corresponding to trajectory A is the mirror reflection with respect s-axis of this solution

with $\sigma = \frac{3\kappa\beta}{u^4}$. The Fig. 1 shows the phase trajectories and the Fig. 2 the possible solution with the non-classical soliton shape named peakon that emerges along the chain for this specific case when the trivial boundary condition is imposed, that consequently leads to the parameter α to have zero value.

3.2 The Case of the Region: $0 < \alpha < \sqrt{\frac{u^4\beta}{3\kappa}}$

In this case, we have four distinguished trajectories, see Figs. 3 and 4. In Fig. 3 we can observe that trajectories that meet the lines $y = \pm u/(3\kappa)^{1/2}$ in the points $(0, \pm u/(3\kappa)^{1/2})$ can be continued in two different ways

On the other hand, the analytical expression for the solution takes the following analytical form.

$$2\varsigma\sqrt{fa}\,(s - s_0) = -\mathrm{Arcsin}\left(1 - f\theta^2\right) \pm \mathrm{Ln}\left(\frac{1 + \omega\theta\sqrt{2f - f^2\theta^2}}{2}\right) \tag{8}$$

Fig. 3 Trajectories of the system (7) with $0 < \alpha < u^2\sqrt{\frac{\beta}{3\kappa}}$

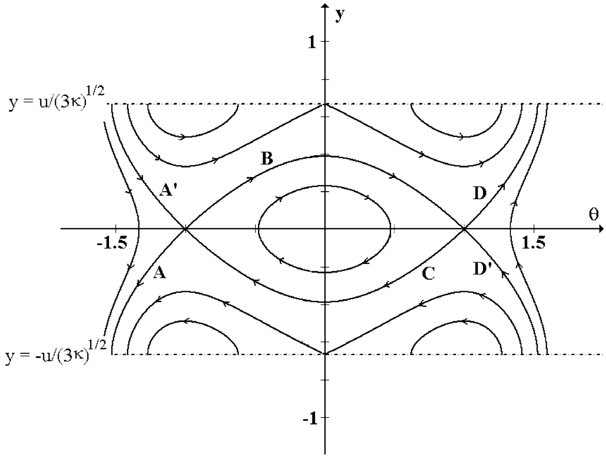
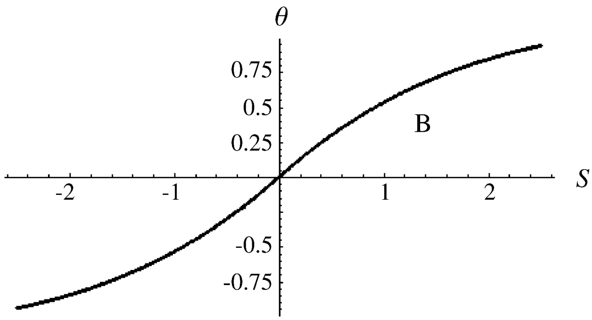


Fig. 4 Numerical representation of the solution corresponding to trajectory B of Fig. 3, the value of parameter is $\alpha = 1$



with $f = \frac{\beta}{\alpha}, a = \frac{2u^2}{3\kappa}$. The $\pm$ sign permits to switch between solutions. Thus if we have the $+$ sign we will obtain kink compactons with singular slope [9] and peak soliton on the step similar to that represented on (2). These solutions correspond to the tagged lines of A and D in the Fig. 3. For the case when the sign is negative $-$, we show numerically that this solution corresponds to the non classical kink compacton. This solution is similar to the famous kink solitons but without tails. In the phase space its trajectory corresponds to the tagged curves with letter B in the Fig. 3. Without loss of generality, we can put here $\alpha = 1$. The solution C is a reflection of this one with respect to the θ axis.

3.3 Case when $\alpha = \sqrt{\frac{u^4\beta}{3\kappa}}$

In this case we are able to construct some trajectories in two different forms, see Figs. 5 and 6 that has a similarity with the solution C. The anti-bubble compact structure is obtained in this case and it is shown in the Fig. 6 by the tagged lines B and B'. Solutions corresponding to trajectories A and D are similar to those of Fig. 2. The analytical expression for the solution is

$$2\sqrt{fa}\,(s - s_0) = -\text{Arcsin}(1 - f\theta^2) + \text{Ln}\left(\frac{\sqrt{f}\theta + \sqrt{2 - f\theta^2}}{2}\right)^2. \tag{9}$$

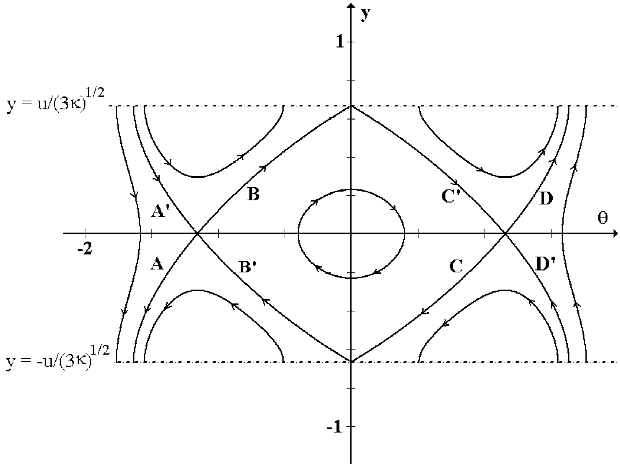


Fig. 5 Trajectories of the system (7) with $\alpha = u^2\sqrt{\frac{\beta}{3\kappa}}$. Trajectory B can be continued by B′ or by C′, analogously with trajectory C

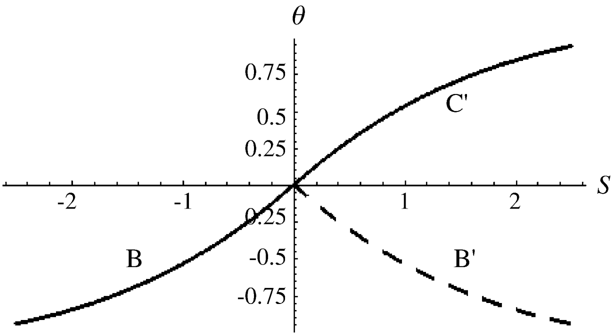


Fig. 6 Numerical solution corresponding to trajectory B of Fig. 5 that can be continued in two different ways. The parameter value taken is $\alpha = 4/3$

Because of the negative field values of this solution, it was named as anti-bubble nonclassical solution.

3.4 Case when $\alpha > \sqrt{\frac{u^4\beta}{3\kappa}}$

This case shows us new solutions that were not presented in the previous paper [9]. The numerical treatment of the nonlinear equation (6) with the parameter value for this case was able to show the compact bubble grey solution that is represented in the Fig. 8. This solution does not have wings, instead it begins and finishes at specifical values of the independent variable. In the phase space, the separations give four trajectories similar to those of the first case. There are others trajectories that can be continued in two different forms, but these always give periodic solutions, that is incompatible with the nontrivial boundary condition applied to the physical real situation, see Figs. 7 up to 9.

The solution B is a mirror reflection of this solution with respect to the s-axis. Solutions corresponding to trajectories A and D are similar to those of (2). As it can be easily checked, there are other solutions that can be continued in two different ways as periodic functions. These solutions correspond to trajectories F and E on the Fig. 7. One way for doing this is starting from trajectory F, directs to trajectory E and the other one is returning back to trajectory F as is shown in Fig. 9.

Fig. 7 Trajectories of the system (7) with $\alpha > u^2\sqrt{\frac{\beta}{3\kappa}}$

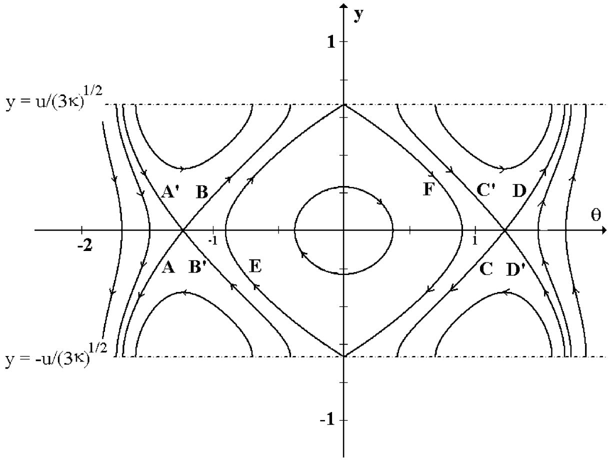


Fig. 8 Numerical representation of the solution corresponding to trajectory C of Fig. 7, for parameter value $\alpha = 1.5$

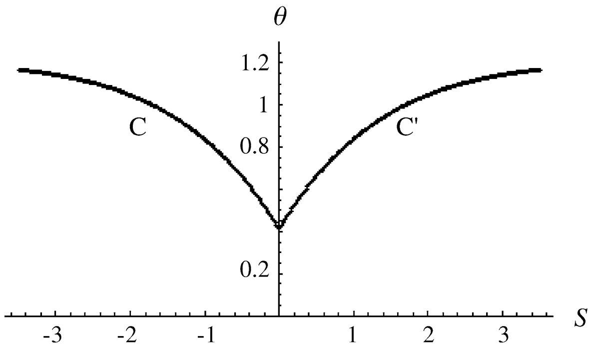
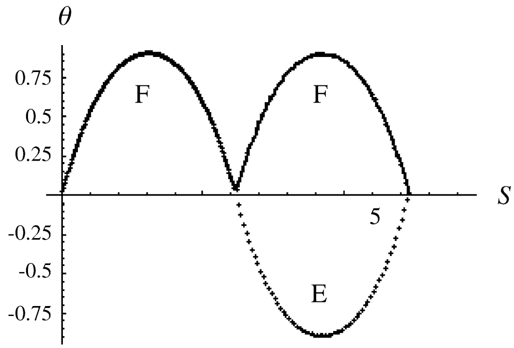


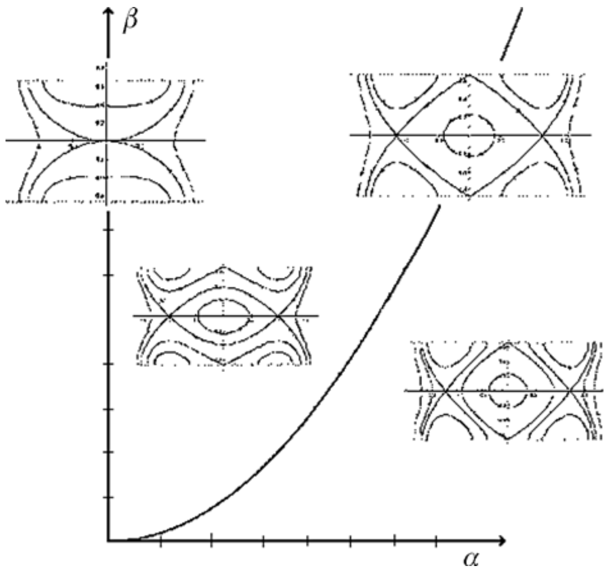
Fig. 9 The solution corresponding to trajectory F of Fig. 7



4 Bifurcations

Physically it is common to treat the system with natural boundary conditions. There are four parameters in the system (6) and the wave speed u is usually not treated as bifurcation parameter, while κ is a constant that is determined by boundary conditions or initial conditions. We consider the physical parameter α as a bifurcation parameter. When α changes the number of equilibria changes too, therefore there could exist some solitary waves and soli-

Fig. 10 Bifurcation diagram in α–β parameter space



ton like structures or nonclassical soliton structures. Thus when $\alpha = 0$, there are only one saddle equilibrium at the origin of the phase space, separatrices are homoclinics trajectories, $\alpha = 0$ is a bifurcation set.

For $0 < \frac{\alpha^2}{\beta} < \frac{u^4}{\sqrt{3\kappa}}$ this equilibrium splits in three equilibria, two saddles and one center, the saddles are joined with heteroclinics trajectories (saddle-saddle connection).

When $\frac{\alpha^2}{\beta} = \frac{u^4}{\sqrt{3\kappa}}$ the equilibria remain as before, but there are another bifurcation: the saddle-saddle connections meet the lines $y = \frac{u}{\sqrt{3\kappa}}$, allowing two different ways for continuing them across those lines (in the cylinder). One of them will be again a saddle-saddle connection and the other become a homoclinic trajectory, just as in the case $\alpha = 0$.

If $\frac{\alpha^2}{\beta} > \frac{u^4}{\sqrt{3\kappa}}$ the saddle-saddle connections disappear, remaining the two saddle equilibria and the center at the origin. As in the case $\alpha = 0$, the separatrices of the saddle equilibria are homoclinic. The Fig. 10 shows the partition of the parameter space $\alpha - \beta$. In this last picture the arc of parabola is given by

$$\alpha^2 = \left(\frac{u^4}{\sqrt{3\kappa}}\right)\beta. \tag{10}$$

The explanation about the partition of the parameter phase space could be done in the following way. Along the line β when $\alpha = 0$ the phase portrait shows one saddle point in the origin. In this case, compact peakon on the step could appear and with saddles it is possible also the appearance of compact bubbles. This is our case, in this parametric line we can observe the appearance of compact solutions and soliton like solution on the step. If we move toward the right direction we encounter the phase space with one center at its origin and two saddles. We can observe that anti-kinks, bubbles anti-bubbles could emerge in this case, which is the same when pass through the parabolic line defined in (10), and reach the sector II. As it can be easily noticed while we are trespassing the line, the systems does undergo parametric phase transitions. In both areas we have similar solutions with different parametric values, (bubbles an kinks) but with different probabilities. We can consider that in this model occur a parametric phase transition of first type. This because at the right and left side of the parabolic curve the critical points saddles and centers emerge at the both regions simultaneously but with different probabilities.

5 Conclusions

The first order parametrical phase transition is observed in the model of particles forming a chain with pure nonlinear interactions. In several cases the homoclinic orbits are truncated and this is the reason of appearing solutions without wings or compacton like solutions. By using the phase portrait analysis we were able to obtain numerically the bubble compacton solution that is no more than a crowdon. This because this solution, represents the propagation of certain type of agglomeration of particles along the chain. Non-classical soliton like solutions are represented in the phase portraits of the Hamiltonian by the connections of a given saddle with itself or with another saddle. The saddles are fixed points having certain stability characteristics determined mainly because we have not here periodic excitation from neighborhood and no dissipative terms. In summary this work complements the previous one that exposed a number of rational nonclassical solutions. In that paper [9] it was found also their energetic discrete values that was quite remarkable. Here it has been shown that the nonclassical solutions was represented in the phase space by trajectories that resembles homoclinic ones, but truncated. The change of critical points when passing from one region to another determines the change of parametric phases. In our case we have observed only the first kind of parametric phase transition.

Acknowledgements This work was supported in part by funds of the CONACYT (Mexico) Sabbatical Research Project and carried out during MAG sabbatical stay at Theoretical Division of Los Alamos National Laboratory.

References

1. Aubry, S.: Physica D **103**, 201 (1997)
2. Flach, S., Willis, C.R.: Phys. Rep. **295**, 181 (1998)
3. Li, Y.A., Olver, P.J., Rosenau, P.: Non-analytic solutions of nonlinear wave models. In: Nonlinear Theory of Generalized Functions, Vienna, 1997. Res. Notes Math., vol. 401, pp. 129–145. Chapman & Hall/CRC, Boca Raton (1999)
4. Kevrekidis, P.G., Konotop, V.V.: Math. Comput. Simul. **62**(1–2), 79–98 (2003)
5. Camassa, R., Holm, D.D.: Phys. Rev. Lett. **71**, 1661–1664 (1993)
6. Rosenau, P., Hyman, J.M.: Phys. Rev. Lett. **70**, 564–567 (1993)
7. Tchofo Dinda, P., Remoissenet, M.: Phys. Rev. E **60**(5), 6218–6221 (1999)
8. Dusuel, S., Michaux, P., Remoissenet, M.: Phys. Rev. E **57**(2), 2320–2326 (1998)
9. de Najera, M.L., Sanchez, J., Aguilar, J.A., Aguero, M.: Int. J. Theor. Phys. **48**, 2098–2111 (2009)
10. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. Springer, Berlin (1983)
11. Agüero, M.A., Paulin, M.J.: Phys. Rev. E **63**, 046606 (2003)